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MÖBIUS TRANSFORMATIONS AND EQUATIONS OF EMPIRICAL CURVES†

P. YA. KOCHINA and O. I. SHISHORINA

Moscow

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A previously proposed method for constructing equations of empirical curves in the mechanics of materials, based on the application of Möbius transformations [1, 2], is amplified. New examples are given for the use of the method in certain filtration problems, taken from Polubarinova-Kochina's book [3].

1. SUPPOSE we are given a curve

$$y = f(x) \tag{1.1}$$

in an interval $0 \leqslant x \leqslant X$, and $f(0) = y_0$; $f(x) = Y$.

Application of the Möbius transformation

$$\xi = x/(X - x), \quad \eta = y/(Y - y) \tag{1.2}$$

transforms Eq. (1.1) into the equation

$$\eta = \Phi(\xi) \tag{1.3}$$

We shall call (1.3) the image of the curves (1.1); the curves (1.1) will be referred to as the pre-image of the curves (1.3).

The transformation (1.2) may be interpreted [1, 2] as "the ratio of what has happened to what still remains to happen".

Applying (1.2) to a considerable variety of curves of the type indicated, one generally obtains straight lines in the ξ, η plane. We shall assume that Eq. (1.3) reduces to the equation of a straight line

$$\eta = A + B\xi \tag{1.4}$$

Substituting expressions (1.2) into this equation and then transforming to dimensionless variables

$$\bar{x} = x/X; \quad \bar{y} = y/Y \tag{1.5}$$

we obtain

$$\bar{y} = \frac{A + (B - A)\bar{x}}{A + 1 + (B - A - 1)\bar{x}} \tag{1.6}$$

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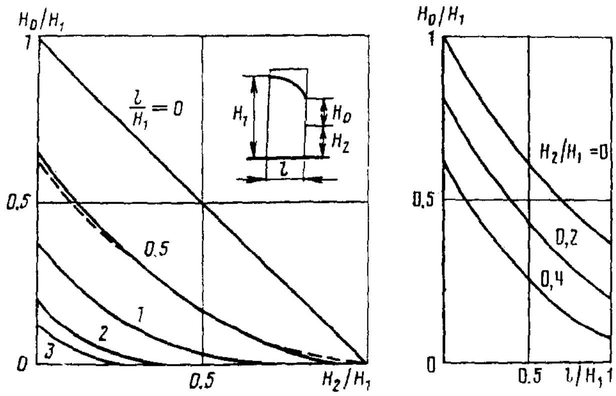

FIG. 1.

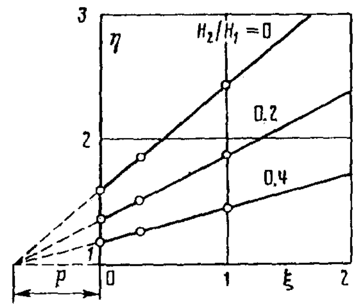

FIG. 2.

that is, we obtain a hyperbola passing through the point (1.1), with two parameters A and B whose geometrical meaning is obvious.

If we consider not one curve (1.1) but a whole family, obtained in some experiment, the result is frequently a pencil of straight lines through a point. Denoting its coordinates by ξ^*, η^* we can write instead of (1.4)

$$\eta - \eta^* = B(\xi - \xi^*) \quad (A = \eta^* - B\xi^*) \tag{1.7}$$

Suppose that $y = y_0$ at $x = 0$. Using (1.7) we can transform Eq. (1.6) to the form

$$\bar{y} = \frac{\bar{y}_0 \xi^* + [\eta^* - (\xi^* + \eta^* + 1)\bar{y}_0]\bar{x}}{\xi^* + [\eta^* - (1+\eta^*)\bar{y}_0^* - \xi^*]\bar{x}}, \qquad \bar{y}_0 = \frac{y_0}{Y} \tag{1.8}$$

This is a hyperbola through the points (0, y_0) and (1.1). Comparing the coefficients of Eqs (1.6) and (1.8), we see that $A = \bar{y}_0\xi^*$, $A + 1 = \xi^*$, so that

$$\xi^* = 1/(1 - \bar{y}_0) = Y/(Y - \bar{y}_0) \tag{1.9}$$

Comparing the coefficients of $\bar{x}$ in (1.6) and (1.8) gives nothing new, and the formula for η^*, besides y_0, Y, also involves B

$$\eta^* = B\xi^* + A = (BY + y_0)/(Y - y_0) \tag{1.10}$$

(Note that the formula for B contains a derivative: $B = Y/(XY')$, $Y' = y'(X)$.)

It is obvious from formulae (1.9) and (1.10) that the result is not always an exact pencil of straight lines. However, as already noted, it is often approximately a pencil, in which case approximate values of ξ^* and η^* may be determined. The images of curves other than hyperbolae in the ξ, η plane are not straight lines, but they are curves with asymptotes which, in the case of curves, do indeed intersect at one point.

Another transformation

$$\xi = (x - x_0)/(X - x), \quad \eta = (y - y_0)/(Y - y) \tag{1.11}$$

sends the point (x_0, y_0) to the origin ($\xi = 0$; $\eta = 0$) and the point (X, Y) to the point at infinity in the ξ, η plane. We may therefore consider the following pencil of straight lines in the ξ, η plane

$$\eta = B\xi \tag{1.12}$$

whose pre-image is the hyperbola

$$y = \frac{(BY - y_0)x + Xy_0 - Bx_0\ Y}{(B - 1)x + X - Bx_0} \tag{1.13}$$

2. We will give some examples.

Example 1. We are given a family of curves [3] relating to calculations of a coffer-dam, i.e. a rectangular earth dam (Fig. 1). When water flows from the headwater to the tailwater, a free surface forms in the body of the dam, in which there will always be an interval of leakage H_0. Accurate theoretical calculations here are extremely complicated; some of the curves have been calculated exactly, others are interpolations.

The basic variable here will be the water level H_1 in the headwater.

Figure 1 plots $y = H_0 / H_1$ as a function of $x = H_2 / H_1$ (where H_2 is the depth of the water in the tailwater) for different values of the quotient l / H_1, where l is the width of the dam.

Here $Y = 0$, so the transformation (1.2) is useless. In (1.11) we can put $x_0 = 0$, $Y = 0$, $X = 1$. We then obtain from (1.13) the following equation with one parameter B

$$y = \frac{y_0(1 - x)}{1 + (B - 1)x}$$

When $B = 1$ one obtains the straight line $y = y_0(1 - x)$. At $y_0 = 0.61$, $B = 2.22$ one obtains the dashed curve in Fig. 1.

Example 2. Figure 2 shows $y = H_0 / H_1$ as a function of $x = l / H_1$ for different ratios H_2 / H_1 for the dam of Example 1. We show some of the curves, up to $l / H_1 = 1$ (coffer-dams are usually narrow, $l / H_1 < 1$). It turned out that the transformation (1.2) yields a pencil of straight lines with centre at the point $\xi = -0.7$, $\eta = 1$. This confirms that the curves of the family in Fig. 2 are approximately hyperbolae.

3. The geometrical value of the parameters A and B has been determined [1, 2, 4],† and various properties of the hyperbolae (1.6) and more general curves have been ascertained. We now present a different derivation of some of these properties, using the concept of the cross-ratio.

The cross-ratio (also known as the anharmonic ratio) of four collinear points A_1, A_2, A_3, A_4 (Fig. 3) is defined as the quotient [5]

$$W = (A_1 A_2 A_3 A_4) = (A_1 A_3 / A_3 A_2) : (A_1 A_4 / A_4 A_2)$$

The cross-ratio is a special case of the [German term] *Wurf*—an anharmonic range of points. For brevity, we shall use this term instead of "cross-ratio".

Setting $A_1 = x_0$, $A_2 = x$, $A_3 = x_1$, $A_4 = X$, we consider the *Wurf* for the abscissae and ordinates of the hyperbola

$$y = (ax + b)/(cx + d) \tag{3.1}$$

an arc of which is shown in Fig. 3

$$\begin{aligned} W_x &= \frac{x_1 - x_0}{x - x_1} : \frac{X - x_0}{x - X} = \frac{(x_1 - x_0)}{(X - x_0)} \frac{(X - x)}{(x_1 - x)} \\ W_y &= \frac{y_1 - y_0}{y - y_1} : \frac{Y - y_0}{y - Y} = \frac{(y_1 - y_0)}{(Y - y_0)} \frac{(Y - y)}{(y_1 - y)} \end{aligned} \tag{3.2}$$

†See also KOCHINA P. Ya., and SHISHORINA O. I., Möbius transformations and their applications. Preprint No. 307, Inst. Probl. Mekhaniki Akad. Nauk SSSR, Moscow, 1987.

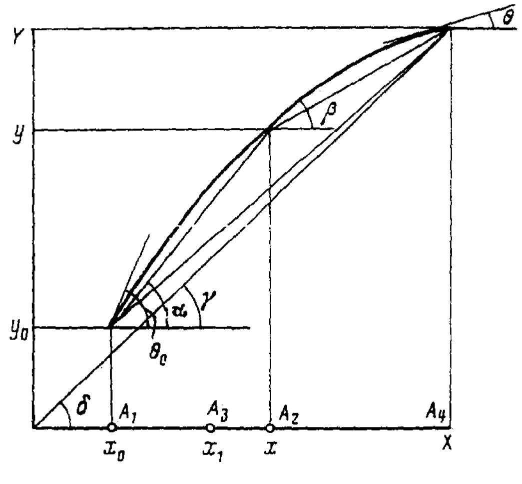

FIG. 3.

In the problems considered here, the point x_1, y_1 is of auxiliary significance; the interesting points are (x_0, y_0), (x, y), (X, Y).

The limit of the quotient of the *Wurf* as $x_1 \to x_0$ (then $y_1 \to y_0$ and the fraction $(y_1 - y_0)/(x_1 - x_0)$ tends to the derivative $f'(x_0)$) is

$$\frac{W_y}{W_x} = [f'(x_0)\frac{Y-y}{X-x}] : \frac{Y-y_0}{X-x_0} \tag{3.3}$$

If $y = f(x)$ is the equation of the hyperbola (3.1), then

$$y' = f(x) = (ad - bc)/(cx+d)^2$$

Substituting

$$Y = \frac{aX+b}{cX+d}, \quad y = \frac{ax+b}{cx+d}, \quad y_0 = \frac{ax_0+b}{cx_0+d}$$

in formulae (.3.3), we find that $W_y = W_x = 1$.

Figure 3 clarifies the geometrical significance of the fractions

$$\frac{y-y_0}{x-x_0} = \operatorname{tg}\alpha, \quad \frac{Y-y}{X-x} = \operatorname{tg}\beta, \quad \frac{Y-y_0}{X-x_0} = \operatorname{tg}\gamma$$

and also demonstrates that $f'(x_0) = \operatorname{tg}\theta_0$, $f'(X) = \operatorname{tg}\theta$. It therefore follows from (3.3) that

$$\operatorname{tg}\theta_0 = \operatorname{tg}\gamma \operatorname{tg}\alpha / \operatorname{tg}\beta \tag{3.4}$$

To get an expression for tg θ —the slope of the tangent to the hyperbola at the other end of the interval A_1A_4, we interchange x and x_1, y and y_1 in (3.2). Letting $x \to X$, $y \to Y$, we finally obtain

$$\operatorname{tg}\theta = \operatorname{tg}\beta \operatorname{tg}\gamma / \operatorname{tg}\alpha \tag{3.5}$$

It follows from (3.4) and (3.5) that

$$\operatorname{tg}\theta_0 \operatorname{tg}\theta = y'(x_0) y'(X) = \operatorname{tg}^2\gamma, \quad \operatorname{tg}\theta_0 / \operatorname{tg}\theta = \operatorname{tg}^2\alpha / \operatorname{tg}^2\beta$$

These relationships enable one to express the slopes of the tangents $\mathrm{tg}\,\theta_0$, $\mathrm{tg}\,\theta$, which are most difficult to determine, in terms of the tangents of the chord angles $\mathrm{tg}\,\alpha$, $\mathrm{tg}\,\beta$, $\mathrm{tg}\,\gamma$.

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